

# NON-MATRIX POLYNOMIAL IDENTITY ENVELOPING ALGEBRAS

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ABSTRACT. Let  $L$  be a restricted Lie superalgebra with its restricted enveloping algebra  $u(L)$  over a field  $\mathbb{F}$  of characteristic  $p > 2$ . A polynomial identity is called non-matrix if it is not satisfied by the algebra of  $2 \times 2$  matrices over  $\mathbb{F}$ . We characterize  $L$  when  $u(L)$  satisfies a non-matrix polynomial identity.

## 1. INTRODUCTION

A variety of associative algebras over a field  $\mathbb{F}$  is called non-matrix if it does not contain  $M_2(\mathbb{F})$ , the algebra of  $2 \times 2$  matrices over  $\mathbb{F}$ . A polynomial identity (PI) is called non-matrix if  $M_2(\mathbb{F})$  does not satisfy this identity. Latyshev in his attempt to solve the Specht problem proved that any non-matrix variety generated by a finitely generated algebra over a field of characteristic zero is finitely based [L77, L80]. The complete solution of the Specht problem in the case of characteristic zero is given by Kemer [K91].

Although several counterexamples are found for the Specht problem in the positive characteristic [AK], the development in this area has lead to some interesting results. Kemer in [K96] investigated the relation between PI-algebras and nil algebras and asked whether the Jacobson radical of a relatively free algebra of countable rank over an infinite field of positive characteristic is a nil ideal of bounded index. Amitsur had already proved in [Am] that the Jacobson radical of a relatively-free algebra of countable rank is nil and Samoilov in [Sam] proved that the Jacobson radical of a relatively free algebra of countable rank over an infinite field of positive characteristic is a nilideal of bounded index. The non-matrix varieties have been further studied in [BRT, MPR, R97].

Enveloping algebras satisfying polynomial identities were first considered by Latyshev [L63] by proving that the universal enveloping algebra of a Lie algebra  $L$  over a field of characteristic zero satisfies a

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PI if and only if  $L$  is abelian. Latyshev's result was extended to positive characteristic by Bahturin [B74]. Passman [P] and Petrogradsky [P91] considered the analogous problem for restricted Lie algebras and their envelopes.

Let  $L = L_0 \oplus L_1$  be a restricted Lie superalgebra with the bracket  $(\ , \ )$ . We denote the restricted enveloping algebra of  $L$  by  $u(L)$ . All algebras in this paper are over a field  $\mathbb{F}$  of characteristic  $p > 2$  unless otherwise stated. In case  $p = 3$  we add the axiom  $((y, y), y) = 0$ , for every  $y \in L_1$ . This identity is necessary to embed  $L$  in  $u(L)$ . Restricted Lie superalgebras whose enveloping algebras satisfy a polynomial identity have been characterized by Petrogradsky [P92]. The purpose of this paper is to characterize restricted Lie superalgebras whose restricted enveloping algebras satisfy a non-matrix PI. Riley and Wilson considered similar conditions for restricted enveloping algebras and group algebras in [RW]. Recall that a subset  $X \subseteq L_0$  is called  $p$ -nilpotent if there exists an integer  $s$  such that  $x^{p^s} = 0$ , for every  $x \in X$ . Our main result is as follows.

**Main Theorem.** *Let  $L = L_0 \oplus L_1$  be a restricted Lie superalgebra over a perfect field and denote by  $M$  the subspace spanned by all  $y \in L_1$  such that  $(y, y)$  is  $p$ -nilpotent. The following statements are equivalent:*

- (1)  $u(L)$  satisfies a non-matrix PI.
- (2)  $u(L)$  satisfies a PI,  $(L_0, L_0)$  is  $p$ -nilpotent,  $\dim L_1/M \leq 1$ ,  $(M, L_1)$  is  $p$ -nilpotent, and  $(L_1, L_0) \subseteq M$ .
- (3) The commutator ideal of  $u(L)$  is nil of bounded index.

Theorem 2.6 below recalls Petrogradsky's characterization of when  $u(L)$  satisfies an arbitrary PI strictly in terms of the underlying Lie superalgebra structure of  $L$ ; this allows one to replace (2) with a similar such characterization (that is too cumbersome to state here). Furthermore, we show that (2) implies (3) over any field. However, given that  $u(L)$  satisfies a non-matrix PI, the restriction on the field is necessary to be able to show that  $\dim L_1/M \leq 1$ . In Section 4, we show that over a non-perfect field there exists a restricted Lie superalgebra  $L = L_0 \oplus L_1$  such that  $\dim L_1 = 2$ , commutator ideal of  $u(L)$  is nil of index  $2p$  and yet  $(y, y)$  is not  $p$ -nilpotent, for every  $y \in L_1$ . This is in complete contrast with the enveloping algebras of ordinary Lie superalgebras satisfying a non-matrix PI, see Theorem 1.2 of [BRU] where a similar characterization does not require any restriction on the field.

## 2. PRELIMINARIES

Unless otherwise stated, all algebras are over a field  $\mathbb{F}$  of characteristic  $p > 2$ . Let  $A = A_0 \oplus A_1$  be a vector space decomposition of a

non-associative algebra over  $\mathbb{F}$ . We say that this is a  $\mathbb{Z}_2$ -grading of  $A$  if  $A_i A_j \subseteq A_{i+j}$ , for every  $i, j \in \mathbb{Z}_2$  with the understanding that the addition  $i+j$  is mod 2. The components  $A_0$  and  $A_1$  are called even and odd parts of  $A$ , respectively. Note that  $A_0$  is a subalgebra of  $A$ . One can associate a Lie super-bracket to  $A$  by defining  $(x, y) = xy - (-1)^{ij}yx$  for every  $x \in A_i$  and  $y \in A_j$ . If  $A$  is associative, then for any  $x \in A_i$ ,  $y \in A_j$  and  $z \in A$  the following identities hold:

- (1)  $(x, y) = -(-1)^{ij}(y, x)$ ,
- (2)  $(x, (y, z)) = ((x, y), z) + (-1)^{ij}(y, (x, z))$ .

The above identities are the defining relations of Lie superalgebras. Furthermore,  $A$  can be viewed as a Lie algebra by the usual Lie bracket  $[u, v] = uv - vu$ .

If  $L$  is a Lie superalgebra, we denote the bracket of  $L$  by  $(,)$ . The adjoint representation of  $L$  is given by  $\text{ad } x : L \rightarrow L$ ,  $\text{ad } x(y) = (y, x)$ , for all  $x, y \in L$ . The notion of restricted Lie superalgebras can be easily formulated as follows:

**Definition 2.1.** A Lie superalgebra  $L = L_0 \oplus L_1$  is called restricted, if there is a  $p$ th power map  $L_0 \rightarrow L_0$ , denoted as  $^{[p]}$ , satisfying

- (a)  $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ , for all  $x \in L_0$  and  $\alpha \in \mathbb{F}$ ,
- (b)  $(y, x^{[p]}) = (y, {}_p x) = (\text{ad } x)^p(y)$ , for all  $x \in L_0$  and  $y \in L$ ,
- (c)  $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$ , for all  $x, y \in L_0$  where  $s_i$  is the coefficient of  $\lambda^{i-1}$  in  $(\text{ad } (\lambda x + y))^{p-1}(x)$ .

For example, every  $\mathbb{Z}_2$ -graded associative algebra inherits a restricted Lie superalgebra structure.

Let  $L$  be a restricted Lie superalgebra. We denote the (restricted) enveloping algebra of  $L$  by  $u(L)$ . The augmentation ideal  $\omega(L)$  is the ideal of  $u(L)$  generated by  $L$ . The analogue of the PBW Theorem is as follows. We refer to [BMPZ] for basic background.

**Theorem 2.2.** Let  $L = L_0 \oplus L_1$  be a restricted Lie superalgebra and let  $\mathcal{B}$  be a totally ordered basis for  $L$  consisting of  $\mathbb{Z}_2$ -homogeneous elements. Then  $u(L)$  has a basis consisting of PBW monomials, that is, monomials of the form  $x_1^{a_1} \dots x_s^{a_s}$  where  $x_1 < \dots < x_s$  in  $\mathcal{B}$ ,  $0 \leq a_i < p$  whenever  $x_i \in L_0$ , and  $0 \leq a_i \leq 1$  whenever  $x_i \in L_1$ .

Since  $L$  embeds into  $u(L)$  we identify  $x^{[p]}$  with  $x^p$ , for every  $x \in L_0$ . Note that  $u(L)$  can be viewed as a Lie algebra via the Lie bracket  $[x, y] = xy - yx$  and if  $x \in L_0$  and  $y \in L$  then the bracket  $(x, y)$  in  $L$  is the same as the bracket  $[x, y]$  in  $u(L)$ . Let  $H$  be a subalgebra of  $L$ . We denote by  $H'$  the commutator subalgebra of  $H$ , that is  $H' = (H, H)$ . For a subset  $X \subseteq L$ , we denote by  $\langle X \rangle_p$  or  $X_p$  the restricted ideal of  $L$

generated by  $X$ . Also,  $\langle X \rangle_{\mathbb{F}}$  denotes the subspace spanned by  $X$ . An element  $x \in L_0$  is called *p-nilpotent* if there exists some non-negative integer  $t$  such that  $x^{p^t} = 0$ . Also, recall that  $X$  is said to be *p-nil* if every element  $x \in X$  is *p-nilpotent* and  $X$  is *p-nilpotent* if there exists a positive integer  $k$  such that  $x^{p^k} = 0$ , for every  $x \in X$ . By an ideal of  $L$  we always mean a restricted ideal, that is  $I$  is an ideal of  $L$  if  $(I, L) \subseteq I$  and  $I_0$  is closed under the  $p$ -map.

Let  $B$  and  $C$  be subspaces of  $L$ . We denote by  $(B, C)$  the subspace spanned by all commutators  $(b, c)$ , where  $b \in B$  and  $c \in C$ . The lower Lie central series of  $L$  is defined by setting  $\gamma_1(L) = L$  and  $\gamma_n(L) = (\gamma_{n-1}(L), L)$ , for every  $n \geq 2$ . Recall that  $L$  is called nilpotent if  $\gamma_n(L) = 0$ , for some  $n$ . The derived subalgebra of  $L$  is defined by setting  $\delta_0(L) = L$  and  $\delta_{i+1}(L) = (\delta_i(L), \delta_i(L))$ , for every  $i \geq 0$ . Also,  $L$  is called solvable if  $\delta_m(L) = 0$ , for some  $m$ , and the least of such  $m$  is called the derived length of  $L$ . Moreover, long commutators are left tapped, that is  $(x, y, x) = ((x, y), z)$ .

Note that Engel's Theorem holds for Lie superalgebras, see [Sch], for example.

**Theorem 2.3** (Engel's Theorem). *Let  $L$  be a finite-dimensional Lie superalgebra such that  $\text{ad } x$  is nilpotent, for every homogeneous element  $x \in L$ . Then  $L$  is nilpotent.*

The proof of the following lemma follows from Engel's Theorem and the fact that  $(\text{ad } x)^2 = \frac{1}{2}\text{ad } (x, x)$ , for every  $x \in L_1$ .

**Lemma 2.4.** *Let  $L = L_0 \oplus L_1$  be a finite-dimensional restricted Lie superalgebra. If  $L_0$  is *p-nil* then  $L$  is nilpotent.*

**Lemma 2.5.** *Let  $L$  be a restricted Lie superalgebra. Then  $\omega(L)$  is associative nilpotent if and only if  $L$  is finite-dimensional and  $L_0$  is *p-nil*.*

*Proof.* The if part follows from the PBW Theorem. We prove the converse by induction on  $\dim L$ . By Lemma 2.4,  $L$  is nilpotent and so there exists a non-zero element  $z$  in the center  $Z(L)$  of  $L$ . Since  $Z(L)$  is homogeneous we may assume that either  $z \in L_0$  or  $z \in L_1$ . If  $z \in L_1$  then  $z^2 = (z, z) = 0$ . If  $z \in L_0$  then we can replace  $z$  with its  $p$ -powers so that  $z^p = 0$ . So in either case  $z^p = 0$  in  $u(L)$ . Now consider  $H = L/\langle z \rangle_p$ . Then by induction hypothesis  $\omega(H)$  is nilpotent. This means that  $\omega^m(L) \subseteq \langle z \rangle_p u(L)$ , for some  $m$ . It then follows that  $\omega^{mp}(L) \subseteq \langle z^p \rangle_p u(L) = 0$ , as required.  $\square$

We shall use the following two results.

**Theorem 2.6** ([P92]). *Let  $L = L_0 \oplus L_1$  be a restricted Lie superalgebra. Then  $u(L)$  satisfies a PI if and only if there exist homogeneous restricted ideals  $B \subseteq A \subseteq L$  such that*

- (1)  $L/A$  and  $B$  are both finite-dimensional.
- (2)  $A' \subseteq B$ ,  $B' = 0$ .
- (3) The restricted Lie subalgebra  $B_0$  is  $p$ -nilpotent.

**Proposition 2.7** ([RW]). *Let  $R$  be an associative algebra that satisfies a non-matrix PI over a field of positive characteristic  $p$ . Then there exists an integer  $t$  such that  $R$  satisfies the identity  $([u, v]w)^{p^t} = 0$ .*

### 3. PROOFS

**Proposition 3.1.** *Let  $L$  be a restricted Lie superalgebra over a perfect field  $\mathbb{F}$ . Let  $M$  be the set consisting of all  $y \in L_1$  such that  $(y, y)$  is  $p$ -nilpotent. If  $u(L)$  satisfies a non-matrix PI then the following conditions are satisfied:*

- (1)  $M$  is a subspace of  $L_1$  and  $(L_1, L_0) \subseteq M$ ;
- (2)  $\dim L_1/M \leq 1$ ;
- (3)  $(L_0, L_0)$  and  $(M, L_1)$  are both  $p$ -nilpotent.

*Proof.* Let  $R = [u(L), u(L)]u(L)$ . Proposition 2.7 implies that  $R$  is nil. Let  $y \in M$  and  $x \in L_1$ . Note that  $(x, y) = 2yx$  modulo  $R$ . So,  $(y, x)^{2m} = (y, y)^m(x, x)^m$  modulo  $R$ . Since  $(y, y)$  is  $p$ -nilpotent and  $R$  is nil, we deduce that  $(x, y)$  is  $p$ -nilpotent. So,  $(M, L_1)$  is  $p$ -nil. Let  $y_1, y_2 \in M$  and set  $y = \alpha y_1 + \beta y_2$ , where  $\alpha, \beta \in \mathbb{F}$ . Note that  $(y, y)^{p^i} = (y, \alpha y_1)^{p^i} + (y, \beta y_2)^{p^i}$  modulo  $R$ , for every  $i$ . Since  $(M, L_1)$  and  $R$  are both nil, we deduce that  $(y, y)$  is  $p$ -nilpotent. This proves that  $M$  is a subspace. Since  $[L_1, L_0] \subseteq R$ , we know by Proposition 2.7 that  $[L_1, L_0]$  is nil. Thus, if  $y \in [L_1, L_0]$  then  $(y, y) = 2y^2$  is  $p$ -nilpotent. Hence,  $[L_1, L_0] \subseteq M$  and this finishes the proof of (1).

Now, we show that  $\dim L_1/M \leq 1$ . Let  $I$  be the subset of  $L_0$  consisting of  $p$ -nilpotent elements. By Proposition 2.7,  $I$  is a restricted ideal of  $L_0$  and  $(L_0, L_0) \subseteq I$ . We have also proved that  $(M, L_1) \subseteq I$ . Note that  $I + M$  is a restricted ideal of  $L$ . Without loss of generality, we can replace  $L$  with  $L/(I + M)$ . So,  $(y, y)$  is not  $p$ -nilpotent, for every  $y \in L_1$ . Let  $y, z \in L_1$ . Since  $(L_0, L) = 0$ , we have

$$[y, z] = -(y, z) + 2yz, \quad [y, z]^2 = (y, z)^2 - (y, y)(z, z).$$

By Proposition 2.7, there exists  $m$  such that  $[y, z]^{2p^m} = 0$ . So,

$$(1) \quad (y, z)^{2p^m} = (y, y)^{p^m}(z, z)^{p^m}.$$

Thus, by the PBW theorem,  $(y, y)^{p^m}$  and  $(z, z)^{p^m}$  must be linearly dependent. So,  $(z, z)^{p^m} = \beta(y, y)^{p^m}$ , for some  $\beta \in \mathbb{F}$ . Equation (1) then implies that

$$(2) \quad ((y, z)^{p^m})^2 = \beta((y, y)^{p^m})^2.$$

Using the PBW Theorem again, we deduce that  $(y, y)^{p^m} = \alpha(y, z)^{p^m}$ , for some  $\alpha \in \mathbb{F}$ . Equation (2) implies that  $\beta\alpha^2 = 1$ . So, we must have  $(y, y)^{p^m} = \alpha(y, z)^{p^m}$  and  $(z, z)^{p^m} = \alpha^{-1}(y, z)^{p^m}$ . Let  $\gamma \in \mathbb{F}$  so that  $\gamma^{p^m} = \alpha$ . We have,  $(y, y - \gamma z)^{p^m} = 0$  and  $(z, z - \gamma^{-1}y)^{p^m} = 0$ . So,  $(y - \gamma z, y - \gamma z)^{p^m} = 0$  which implies that  $y - \gamma z = 0$ . Thus,  $y$  and  $z$  are dependent, as required. This proves (2). In order to prove (3), it suffices to show that there exists an integer  $m$  such that  $((L_0, L_0) + (M, L_1))^{p^m} = 0$ . Note that, by Proposition 2.7, there exists an integer  $t$  such that  $[u, v]^{p^t} = 0$ , for all  $u, v \in u(L)$ . By Theorem 2.6, there exists a homogeneous ideal  $A$  of  $L$  of finite codimension such that  $B = \langle A' \rangle_p$  is finite dimensional and  $B_0$  is  $p$ -nilpotent. We can replace  $L$  with  $L/B$ . So we can assume that  $A' = 0$ . In particular,  $A_1 \subseteq M$ . We claim that  $L$  is solvable. Indeed, let  $H = (L_0 + M)/A$ . It follows from Theorem 2.3 that  $H'$  is nilpotent. Thus,  $H$  is solvable and, since  $A$  is abelian, we deduce that  $L_0 + M$  is solvable. But  $L/(L_0 + M)$  is abelian which implies that  $L$  is solvable, as claimed. Now, we argue by induction on the derived length  $s$  of  $L$ . Suppose first that  $L$  is metabelian, that is  $(L', L') = 0$ . Since  $(L_0, L_0)$  is abelian, it follows that  $u^{p^t} = 0$ , for every  $u \in (L_0, L_0)$ . So,  $(L_0, L_0)$  is  $p$ -nilpotent. Now we show that  $(M, L_1)$  is  $p$ -nilpotent. Let  $y_1, \dots, y_n$  be linearly independent elements in  $M$  so that their images span  $M/A_1$ . If  $\dim L_1/M = 1$ , we take  $z \in L_1 \setminus M$ . In part (1), we proved that  $(M, L_1)$  is  $p$ -nil. Thus, there exists an integer  $m > t$  such that  $(y_i, y_j)^{p^m} = (z, y_k)^{p^m} = 0$ , for all  $1 \leq i, j, k \leq n$ . Then

$$\begin{aligned} \left( \sum_{i=1}^n \alpha_i y_i, \beta z + \sum_{j=1}^n \beta_j y_j \right)^{p^m} &= \left( \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j (y_i, y_j) + \sum_{i=1}^n \alpha_i \beta (y_i, z) \right)^{p^m} \\ &= \sum_{i=1}^n \sum_{j=1}^n (\alpha_i \beta_j y_i, y_j)^{p^m} + \sum_{i=1}^n (\alpha_i \beta y_i, z)^{p^m} = 0, \end{aligned}$$

where  $\beta$  and the  $\alpha_i$  and  $\beta_j$  are in  $\mathbb{F}$ . On the other hand, let  $y \in L_1$  and  $x \in A_1$ . Since  $x^2 = 0$ , we get

$$[x, y]^2 = (x, y)^2 - 2(x, y, y)x.$$

Since  $A' = 0$ , we deduce that  $[x, y]^{2p} = (x, y)^{2p}$ . Thus,  $(x, y)^{2p^t} = [x, y]^{2p^t} = 0$ . We deduce that  $(M, L_1)^{p^m} = 0$ . Since  $L'$  is abelian,

it is clear that  $((L_0, L_0) + (M, L_1))^{p^m} = 0$ . Now suppose  $\delta_s(L) = 0$ , for some  $s \geq 3$ . Let  $H = \langle \delta_{s-2}(L) \rangle_p$  and  $K = \langle \delta_{s-1}(L) \rangle_p$ . Note that  $H_1 \subseteq [L_1, L_0] \subseteq M$ . Since  $H$  is a metabelian ideal of  $L$ , we have  $((H_0, H_0) + (H_1, H_1))^{p^m} = 0$ . On the other hand, by the induction hypothesis applied to  $L/K$ , we have  $((L_0, L_0) + (M, L_1))^{p^m} \subseteq K_0$ . But  $K_0 = \langle (H_0, H_0) + (H_1, H_1) \rangle_p$ . So, we get  $((L_0, L_0) + (M, L_1))^{p^{2m}} = 0$ , as required.  $\square$

Note that the difference between the restricted case and ordinary case mentioned in the introduction arises from Equation (1). In the ordinary Lie superalgebra case, discussed in [BRU], Equation (1) immediately implies that  $(y, y) = \alpha(y, z)$  and  $(z, z) = \alpha^{-1}(y, z)$ , for some  $\alpha \in \mathbb{F}$ . We then deduce that  $y$  and  $z$  must be dependent and there is no need for the field to be perfect.

**Lemma 3.2.** *Let  $L$  be a restricted Lie superalgebra and  $A$  a homogeneous ideal of  $L$  of finite codimension. Let  $I$  be an ideal of  $u(A)$  that is stable under the adjoint action of  $L$ . If  $u(L)$  is PI and  $I$  is nil of bounded index then so is  $Iu(L)$ .*

*Proof.* Let  $R = u(L)$ . Let  $x_1, \dots, x_n \in L_0$  and  $y_1, \dots, y_m \in L_1$  such that

$$L = A + \langle x_1, \dots, x_n, y_1, \dots, y_m \rangle_{\mathbb{F}}.$$

We assume that the  $x_i$  and  $y_j$  are linearly independent modulo  $A$ . By the PBW Theorem,  $R$  has a basis consisting of the monomials of the form

$$x_1^{\alpha_1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_m^{\beta_m} w$$

$$0 \leq \alpha_i < p, \quad \beta_j \in \{0, 1\},$$

where the  $w$ 's are PBW monomials in  $u(A)$ . Let  $D = u(A)$ . Note that  $R$  is a right  $D$ -module of finite rank  $r = p^n 2^m$ . Now consider the regular representation

$$\rho : R \rightarrow \text{End}(R_D) = M_r(D),$$

where  $\rho(u) : R \rightarrow R$  is defined by  $\rho(u)v = uv$ , for every  $u, v \in R$ . Note that  $\rho$  is injective because  $R$  is unital. Thus, under  $\rho$ , we can embed  $R$  into  $M_r(D)$ . Since  $I$  is stable under the adjoint action of  $L$ , we have  $RI = IR = RIR$ . We claim that  $RI$  embeds into  $M_r(I)$ . Indeed, let  $v_1 \in RI$  and  $v_2 \in R$ . Since  $\rho(v_1)(v_2) = v_1 v_2 \in RI$ , we can write  $v_1 v_2$  as a linear combinations of elements of the form  $ua$ , where  $u \in R$  and  $a \in I$ . So, each  $ua$  and hence  $v_1 v_2$  is a linear combination of elements of the form  $x_1^{\alpha_1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_m^{\beta_m} b$ , where  $b \in I$ , as claimed.

Therefore, it suffices to show that  $M_r(I)$  is nil of bounded index. Recall Levitzki's Theorem and Shirshov's Height Theorem stating that every  $t$ -generated PI algebra which is nil of bounded index  $s$  is (associative) nilpotent of a bound given as a function of  $s$ ,  $t$ , and  $d$ , where  $d$  is the degree of the polynomial identity, see [Lev] and [Sh]. So, if  $S$  is any  $r^2$ -generated subalgebra of  $I$  then there exists a constant  $k$  such that  $S^k = 0$ . Now, let  $T \in M_r(I)$  and denote by  $S$  the subalgebra of  $I$  generated by all entries of  $T$ . So,  $T^i \in M_r(S^i)$ , for every  $i$ . Since  $S^k = 0$ , we get  $T^k = 0$ . Since  $k$  is independent of  $T$ , it follows that  $M_r(I)$  is nil of bounded index, as required.  $\square$

*Proof of the Main Theorem.* The implication (3)  $\Rightarrow$  (1) is obvious while (1)  $\Rightarrow$  (2) follows from Proposition 3.1. To prove (2)  $\Rightarrow$  (3), we shall use the fact that the class of nil algebras of bounded index is closed under extensions. If  $\dim L_1/M = 1$  then we may assume that  $L_1 = M + \mathbb{F}z$ , where  $(z, z)$  is not  $p$ -nilpotent. By Theorem 2.6, there exists a homogeneous restricted ideal  $A$  of finite codimension in  $L$  such that  $A'$  is finite-dimensional and  $(A')_0$  is  $p$ -nilpotent. Note that, by Lemma 2.4,  $(A, A)$  is nilpotent. Thus,  $\omega(\langle A' \rangle_p)$  is associative nilpotent, by Lemma 2.5. Hence,  $A'u(L)$  is associative nilpotent. So, we can replace  $L$  with  $L/(A')_p$  to assume that  $A$  is abelian. We claim that  $(A_1, L_1)$  is  $p$ -nilpotent. If  $M = L_1$ , then  $(L_1, L_1)$  is  $p$ -nilpotent and the claim is obvious. So we may assume that  $(z, z)$  is not  $p$ -nilpotent and prove  $A_1 \subseteq M$ . Suppose that there exists  $x \in A_1$  such that  $x \notin M$ . Since  $\dim L/M = 1$ , we may assume that  $x = z + \alpha y$ , for some  $y \in M$  and  $\alpha \in \mathbb{F}$ . Then, since  $(x, x) = 0$ , we get

$$(z, z) = (x - \alpha y, x - \alpha y) = -2\alpha(x, y) + \alpha^2(y, y) = (-2\alpha x + \alpha^2 y, y).$$

Since  $y \in M$ , it follows from the hypothesis that  $(y, L_1)$  is  $p$ -nilpotent. So,  $(z, z)$  must be  $p$ -nilpotent, which is a contradiction. Note that  $u(A)$  is the tensor product of a commutative algebra with the Grassmann algebra. Thus,  $u(A)$  satisfies  $[x, y, z] = 0$ . So, we have  $(x+y)^p = x^p + y^p$ , for all  $x, y \in u(A)$ . Let  $B = \langle (A, L) \rangle_p$ . Note that  $B$  is an abelian ideal of  $L$  and since  $(A_0, L_0)$  and  $(A_1, L_1)$  are both  $p$ -nilpotent,  $\omega(B)$  is nil of bounded index. It follows that  $I = Bu(A)$  is nil of bounded index. Furthermore, by Lemma 3.2,  $Iu(L) = Bu(L)$  is also a nil ideal of  $u(L)$  of bounded index. But  $Iu(L)$  is the kernel of the homomorphism  $R \rightarrow u(L/B)$ . Thus, we can replace  $L$  with  $L/B$  to assume that  $A$  is central in  $L$ . It follows that  $L'$  is finite-dimensional. Note that  $A_1u(A)$  is nil of index  $p$ . Since  $A_1$  is a central ideal of  $L$ ,  $A_1u(L)$  is nil of bounded index, by Lemma 3.2. Thus, we may assume that  $A_1 = 0$ . It follows that  $M$  is finite-dimensional. Thus, by Lemma



2.5,  $((M, L_1) + M)u(L)$  is associative nilpotent. We can now assume  $M = 0$ . Hence,  $L_1 = \mathbb{F}z$  and  $(z, L_0) = 0$ . Let  $H = L_0$ . Since  $H'$  is finite-dimensional and  $p$ -nilpotent, we deduce, by Lemma 2.5, that  $\omega(\langle H' \rangle_p)$  is associative nilpotent. So we can replace  $L$  with  $L/\langle H' \rangle_p$ . Hence,  $(L_0, L) = 0$ . It then follows that  $[R, R] = 0$ , as required.  $\square$

#### 4. EXAMPLES

We provide examples showing that the restriction on the field in the main result is necessary.

**Example 4.1.** Let  $L = L_0 \oplus L_1$  be a restricted Lie superalgebra over a non-perfect field  $\mathbb{F}$ , where  $L_0 = \langle x_1, x_2, x_3 \rangle_{\mathbb{F}}$  and  $L_1 = \langle y, z \rangle_{\mathbb{F}}$ . We assume that  $(L_0, L) = 0$  and set  $x_1 = (y, y)$ ,  $x_2 = (z, z)$ , and  $x_3 = (y, z)$ . Let  $\alpha \in \mathbb{F}$  be an element whose  $p$ th root does not lie in  $\mathbb{F}$ . We define the  $p$ -mapping by setting  $x_1^p = x_1$ ,  $x_2^p = \alpha^2 x_1$ , and  $x_3^p = \alpha x_1$ . The following statements hold:

- (1) The commutator ideal of  $u(L)$  is nil of bounded index; hence  $u(L)$  satisfies a non-matrix PI.
- (2)  $(c, c)$  is not  $p$ -nilpotent, for every  $c \in L_1$ .

*Proof.* Note that  $[y, z]^2 = x_3^2 - x_1 x_2$  is a central element in  $u(L)$  and  $[y, z]^{2p} = 0$ . Every element in the commutator ideal of  $u(L)$  is of the form  $u = (\alpha y + \beta z + \gamma)[y, z]$ , where  $\alpha, \beta, \gamma$  are in the center of  $u(L)$ . We observe that

$$\begin{aligned} u^2 &= (\alpha y + \beta z + \gamma)[y, z](\alpha y + \beta z + \gamma)[y, z] \\ &= (\alpha y + \beta z + \gamma)^2 [y, z]^2 + (\alpha y + \beta z + \gamma)[y, z, \alpha y + \beta z + \gamma][y, z] \\ &= (\alpha y + \beta z + \gamma)^2 [y, z]^2 + (\alpha y + \beta z + \gamma)(-2\alpha y - 2\beta z)[y, z]^2 \\ &= (\alpha y + \beta z + \gamma)(-\alpha y - \beta z + \gamma)[y, z]^2 \end{aligned}$$

Thus  $u^{2p} = 0$ , for every  $u \in [u(L), u(L)]u(L)$ . Hence, the commutator ideal of  $u(L)$  is nil of index  $2p$ .

Next we prove (2). Suppose to the contrary that there exists  $c \in L_1$  such that  $(c, c)$  is  $p$ -nilpotent. Without loss of generality, we may assume  $c = y + \beta z$ . We have

$$[c, z] = -(c, z) + 2cz, \quad [c, z]^2 = (c, z)^2 - (c, c)(z, z).$$

Since  $[c, z]^{2p} = 0$ , we get  $(c, z)^{2p} = (c, c)^p (z, z)^p$ . Since  $(c, c)$  is  $p$ -nilpotent,  $(c, z)$  must be  $p$ -nilpotent. So there exists an integer  $m$  such

that  $(y + \beta z, z)^{p^m} = 0$ . Note that

$$\begin{aligned} (y + \beta z, z)^{p^m} &= x_3^{p^m} + \beta^{p^m} x_2^{p^m} = (\alpha x_1)^{p^{m-1}} + \beta^{p^m} (\alpha^2 x_1)^{p^{m-1}} \\ &= \alpha^{p^{m-1}} x_1 + \beta^{p^m} \alpha^{2p^{m-1}} x_1 = 0 \end{aligned}$$

We get  $\beta^{p^m} \alpha^{p^{m-1}} + 1 = 0$ . Hence,  $(\beta^p \alpha + 1)^{p^{m-1}} = 0$  which implies that  $\alpha = (-\beta)^p$ . So  $\alpha$  must have  $p$ th root in  $\mathbb{F}$  which is a contradiction.  $\square$

One might ask if similar examples exist when  $\dim L_1 \geq 3$ . The answer is yes as the following example shows.

**Example 4.2.** Let  $\mathbb{F} \subseteq K$  be a field extension and suppose that there exist  $u, v \in K$  so that  $\{1, u, v\}$  is linearly independent over  $\mathbb{F}$  and  $u^p, v^p \in \mathbb{F}$ . For example, one can take  $K = \mathbb{F}_p(u, v)$  and  $\mathbb{F} = \mathbb{F}_p(u^p, v^p)$ , where  $\mathbb{F}_p$  is the prime field. Indeed, here we have  $[K : \mathbb{F}] = p^2$ .

We define a restricted Lie superalgebra  $L = L_0 \oplus L_1$  over  $\mathbb{F}$  as follows. Let  $L_1 = \langle y_1, y_2, y_3 \rangle_{\mathbb{F}}$ ,  $L_0 = \langle z_{ij}, x_k \mid 1 \leq i < j \leq 3, 1 \leq k \leq 3 \rangle_{\mathbb{F}}$  and assume that  $(L_0, L) = 0$ . Set  $z_{ij} = (y_i, y_j)$ ,  $x_k = (y_k, y_k)$ , for all  $1 \leq i < j \leq 3$  and  $1 \leq k \leq 3$ . Let  $\alpha = u^p$ ,  $\beta = v^p$  and define the  $p$ -mapping by  $x_1^p = x_1$ ,  $x_2^p = \alpha^2 x_1$ ,  $x_3^p = \beta^2 x_1$ ,  $z_{12}^p = \alpha x_1$ ,  $z_{13}^p = \beta x_1$ , and  $z_{23}^p = \alpha \beta x_1$ . The following statements hold:

- (1) The commutator ideal of  $u(L)$  is nil of bounded index.
- (2)  $(c, c)$  is not  $p$ -nilpotent, for every  $c \in L_1$ .

*Proof.* Let  $J$  be the Jacobson radical of  $u(L)$ . So,  $J$  is associative nilpotent. Let  $A = L_0 \oplus \langle y_1, y_2 \rangle$  and set  $P = u(A)$ . Then, by Example 4.1,  $[P, P]P$  is a nil ideal of  $P$  of bounded index. So, by Lemma 3.2, we deduce that  $[P, P]u(L)$  is nil of bounded index. Hence,  $[P, P]u(L) \subseteq J$ . We can similarly prove that  $[y_1, y_3], [y_2, y_3] \in J$ . This implies that the commutator ideal of  $u(L)$  is nil of bounded index. This proves (1).

To prove (2), suppose to the contrary that there exists  $c \in L_1$  such that  $(c, c)$  is  $p$ -nilpotent. Without loss of generality, we may assume  $c = y_3 + \alpha_1 y_1 + \alpha_2 y_2$ . Then since the commutator ideal of  $u(L)$  is nil of bounded index,  $(y_3 + \alpha_1 y_1 + \alpha_2 y_2, y_3)$  is  $p$ -nilpotent. So there exists  $m$  such that  $(y_3 + \alpha_1 y_1 + \alpha_2 y_2, y_3)^{p^m} = 0$ . Note that

$$\begin{aligned} (y_3 + \alpha_1 y_1 + \alpha_2 y_2, y_3)^{p^m} &= x_3^{p^m} + \alpha_1^{p^m} z_{13}^{p^m} + \alpha_2^{p^m} z_{23}^{p^m} \\ &= (\beta^2 x_1)^{p^{m-1}} + \alpha_1^{p^m} (\beta x_1)^{p^{m-1}} + \alpha_2^{p^m} (\alpha \beta x_1)^{p^{m-1}} \\ &= \beta^{p^{m-1}} x \left( \beta^{p^{m-1}} + \alpha_1^{p^m} + \alpha_2^{p^m} \alpha^{p^{m-1}} \right) = 0. \end{aligned}$$

We get  $(\beta + \alpha_1^p + \alpha_2^p \alpha)^{p^{m-1}} = 0$ . Thus,  $\beta + \alpha_1^p + \alpha_2^p \alpha = 0$  and so  $v + \alpha_1 + \alpha_2 u = 0$ . But this means that  $u, v, 1$  are linearly dependent over  $\mathbb{F}$ , a contradiction.

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